First semestral examination 2011
B.Math. (Hons.) IInd year
Algebra III — B.Sury
December 5, 2011 — 10 AM - 1 PM
Attempt only FIVE questions.
Any score of more than 80 will be equated to 80.
Be Brief!

Q 1. (16 marks)
Prove that both the polynomials X^3+X+1 and X^3+X^2+1 are irreducible over $\mathbf{Z}_2$. Further, prove that the two fields $\mathbf{Z}_2[X]/(X^3+X+1)$ and $\mathbf{Z}_2[X]/(X^3+X^2+1)$ are isomorphic.

OR

Determine all $c \in \mathbf{Z}_3$ such that $\mathbf{Z}_3[X]/(X^3+X^2+cX+1)$ is a field.

Q 2. (18 marks)
Let A be a commutative ring with unity. If I is an ideal which is maximal with respect to the property of not being principal, prove that I is a prime ideal. Further, if I is such an ideal, prove that A/I is a principal ideal ring.

OR

Let A be a commutative ring with unity, I be an ideal and $P_1, \cdots, P_m$ be prime ideals such that $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_m$. Then show that $I \subseteq P_i$ for some i.

Q 3. (17 marks)
Prove that every ideal of $\mathbf{Z}[i][X]$ is finitely generated.

OR

Show that the ring $C[0,1]$ is not Noetherian.

Q 4. (17 marks)
If α is an algebraic integer (that is, it is a root of a monic integral polynomial), show that it satisfies a unique monic, irreducible polynomial over $\mathbf{Q}$ and that this polynomial must have coefficients in $\mathbf{Z}$.
Hint: You may use Gauss's lemma.

OR

Show that if $f = \sum_{i=0}^{m} a_i X^i, g = \sum_{j=0}^{n} b_j X^j \in (\mathbf{Z}/1024\mathbf{Z})[X]$ are such that $fg = 0$, then $a_i b_j = 0$ for all i, j.

Q 5. (17 marks)
Prove that the group $\mathbf{Q}^+$ of positive rational numbers is a free abelian group of countably infinite rank.
Hint: Show that the set of primes provides a basis.

OR

Prove that any finitely generated, torsion-free module over a PID is free.

Q 6. (18 marks)
Show that there is no commutative ring A with unity such that $A[X]$ is isomorphic to the ring of integers.
Hint: Show that such an A must be isomorphic as a group to $n\mathbf{Z}$ and derive a contradiction.

OR

If A is an integral domain, and I, J are ideals such that IJ is a principal ideal, prove that I, J are finitely generated.

Q 7. (19 marks)
Prove that X^2+Y^2-1 is irreducible in $K[X,Y]$ for any field K of characteristic different from 2.
Hint: Use Eisenstein's criterion to a suitable UFD.

OR

Let $p \equiv 1$ or $3 \bmod 8$ be a prime. Prove that p is expressible as $x^2 + 2y^2$ for some integers x, y.
Hint: You may assums that $\mathbf{Z}[\sqrt{-2}]$ is a UFD.

Q 8. (18 marks)
Let A be an abelian group and B be a subgroup such that $A/B \cong \mathbf{Z}^n$ for some n. Prove that A is isomorphic as an abelian group with $B \oplus \mathbf{Z}^n$.
Hint: Use the fact that a short exact sequence of modules splits if the quotient is a free module.

OR

Prove that every PID is a UFD and give an example (without proof) of a UFD which is not a PID.

Q 9. (16 marks)
Let A be a local ring with the maximal ideal $\mathbf{m}$. Let M be a finitely generated A-module and $x_1, \cdots, x_n \in M$ be elements such that $M/\mathbf{m}M$ is generated as an $A/\mathbf{m}$-module by the images of the x_i's. Then prove that M is generated by the x_i's.
Hint: You may use the NAK lemma.

OR

Let A be a commutative ring with unity. and M be a finitely generated A-module. If $\theta : M \to M$ is an onto A-module homomorphism, prove that θ is $1-1$ as well.

Q 10. (18 marks)
Let A be a commutative ring with unity.
(i) If I, J are ideals such that there exists an onto A-module homomorphism from A/I to A/J, prove that $I \subseteq J$.
(ii) If an ideal P is free as an A-module, prove that P must be principal.

OR

Let A be an $n \times n$ matrix over an algebraically closed field K. Prove that there is an invertible $n \times n$ matrix P over K such that $PAP^{-1} = A^t$, the transpose of A.
Hint: Use the Jordan form.